Siegel modular form $f \in M_\rho(\Gamma_g)$ — inv. by $\left(\begin{array}{c|c} I & b \\ \hline & I \end{array}\right) \in \Gamma_g$

Fourier expansion $\displaystyle f(\tau) = \sum_{\substack{n \in S_g \\ n \geq 0}} a(n)\, e^{2\pi i \operatorname{Tr}(n\tau)}$

half integral, pos. semidefinite

<u>Defn</u>: f is called <u>singular</u> if $a(n) \neq 0 \Rightarrow n$ singular mat.

<u>Thm</u>: (Freitag, Saldaña, Weissauer) ρ irred. repn. of $GL_g(\mathbb{C})$ w/ highest weight $\lambda_1 \geq \cdots \geq \lambda_g$

$f \in M_\rho(\Gamma_g)$, $f \not\equiv 0$ is singular iff $2\lambda_g < g$.

$f \in S_\rho(\Gamma_g)$ w/ $2\lambda_g < g$. then f is singular.

$a(n)$ supp. on non-singular mat. $a(n)$ supported on sing mat.

$\Rightarrow f = 0$.

No cusp form w/ $2\lambda_g < g$.

$$\operatorname{corank}(\rho) := \#\{1 \le i \le g \mid \lambda_i = \lambda_g\}.$$

$$\operatorname{rank}(f) := \max\{\operatorname{rank}(n) \mid a(n) \neq 0\}.$$

$$\operatorname{corank}(f) := g - \min\{\operatorname{rank}(n) \mid a(n) \neq 0\}$$

VI

<u>Ex</u>: cusp forms rank $= g$.

Siegel Eisenstein series $E_{g,0,k} = \displaystyle\sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in P_0 \backslash \Gamma_g} \det(c\tau + d)^{-k}$

P_0 Siegel parabolic $\left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right)$

$$\operatorname{corank} E_{g,0,k} = g. \qquad (a(0) \neq 0).$$

$\operatorname{corank} f = k.$ $\quad \Phi^{k+1} f = 0.$

$$\Phi f(\tau') = \sum_{n'} a\begin{pmatrix} n' & 0 \\ 0 & 0 \end{pmatrix} e^{2\pi i \operatorname{Tr}(n'\tau')}$$

Theta series in the Siegel setting

$$\varepsilon = \begin{pmatrix} \varepsilon' \\ \varepsilon'' \end{pmatrix} \qquad \varepsilon', \varepsilon'' \in \{0,1\}^g$$

$$\theta[\varepsilon](\tau) := \sum_{m \in \mathbb{Z}^g} \exp\left(2\pi i \left({}^t(m + \tfrac{1}{2}\varepsilon')\tau(m + \tfrac{1}{2}\varepsilon') + \tfrac{1}{2}\,{}^t(m + \tfrac{1}{2}\varepsilon')\varepsilon'' \right)\right)$$

$(\equiv 0$ if ε "odd" $({}^t\varepsilon' \varepsilon''$ odd$)\,)$

In Zagier, these are θ_M, θ_F, $\underset{\substack{|||\\ 0}}{\theta_1}$, θ_2, θ_3, θ_4

$(2^g+1)2^{g-1}$ $\qquad g=1 \qquad$ 3 non-zero θ's.

$\theta[\varepsilon]$ Siegel modular forms of wt $\frac{1}{2}$ level 2.

Ex: $\quad g=1 \qquad \left(\theta\begin{bmatrix}0\\0\end{bmatrix}\theta\begin{bmatrix}0\\1\end{bmatrix}\theta\begin{bmatrix}1\\0\end{bmatrix}\right)^8 = 2^8\Delta \in S_{12}(\Gamma_1)$.

wt $\frac{1}{2}\cdot 3\cdot 8 = 12$

level becomes better level 1

$g = 2$ (10 even ε's)

$$-2^{-14} \prod_{\varepsilon \text{ even}} \theta[\varepsilon]^2$$ wt 10, level 1, cusp form χ_{10}

$$\prod_{\varepsilon \text{ even}} \theta[\varepsilon] \Big(\sum_{\varepsilon_1+\varepsilon_2+\varepsilon_3 \text{ odd}} \pm \theta[\varepsilon_1]\theta[\varepsilon_2]\theta[\varepsilon_3] \Big)^{20}$$ wt 35, level 1, cusp form.

B pos. def even unimodular matrix of size $r \times r$ $r \equiv 0 \mod 8$.

$H_k(r, g)$ space of harmonic polynomials $\sum_{i,j} \frac{\partial^2}{\partial z_{ij}^2} P = 0$

$$P : \mathbb{C}^{r \times g} \longrightarrow \mathbb{C}$$

s.t. $P(zM) = (\det M)^k P(z) \quad \forall M \in GL_g(\mathbb{C})$

$$\rightsquigarrow \theta_{B,P}(\tau) := \sum_{A \in \mathbb{Z}^{r\times g}} P(\sqrt{B}A)\, e^{2\pi i \cdot \frac{1}{2} \mathrm{Tr}({}^tA B A \tau)}$$

Fock model

$$\in M_{k+\frac{r}{2}}(\Gamma_g)$$

weighted sum of θ = Eisenstein series (Siegel-Weil formula)

Fourier-Jacobi expansion

$g=2$ generic Whittaker Fourier coeff. $\leftarrow$ (4)

symplectic gp. odd number has even mult.

4

f inv. by $\left(\begin{array}{c|c} I & b \\ \hline & I \end{array}\right)$ $b \in Sym_2(\mathbb{Z})$ (2^2)

inv. by $\left(\begin{array}{cc|cc} 1 & & b & \\ & 1 & & \\ \hline & & 1 & \\ & & & 1 \end{array}\right)$ $b \in \mathbb{Z}$. $(2\ 1^2)$

trivial $\leftarrow$ (1^4)

$f \in M_k(\Gamma_g)$

$$f(\tau) = \sum \phi_m(\tau'', z)\, e^{2\pi i m \tau'}$$

$$\tau = \left(\begin{array}{c|c} \tau' & z \\ \hline {}^t z & \tau'' \end{array}\right) \begin{array}{cc} 1 & r \\ g-1 & g-r \end{array}$$

$\phi_m(\tau'', z)$ is a Jacobi form on $\mathfrak{h}_{g-1} \times \mathbb{C}^{g-1} \circlearrowleft Sp_{2g-2}(\mathbb{Z}) \times H_{g-1}(\mathbb{Z})$ has some transformation property.

See. Sec. 11.